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Quasi Product on Boolean D-Posets

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Abstract The pointwise multiplication on a full tribe and the product operation on an MV-algebra play a crucial role in the construction of a joint observable. In the present paper we introduce a quasi product operation on Boolean D-posets and describe its properties. Our quasi product generalizes product on MV-algebras and in some cases also t-norms.

Keywords Boolean D-poset · MV-algebra · Quasi product · t-norm

1 Basic notions

D-posets [14] and, equivalently, effect algebras [10] generalize various classical algebraic structures modeling quantum mechanics and the fuzzy sets theory systems. The two structures are based on different approaches. The primary operation on effect algebras is a sum and the primary operation on D-posets is a difference of two comparable elements.

A D-poset is a partially ordered set $\mathcal{P}$, with a greatest element 1 and a smallest element 0, on which a partial binary operation difference $b \ominus a$ is defined if and only if $a \leq b$; it fulfills the following conditions:

(D1) $b \ominus 0 = b$;
(D2) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

In every D-poset $\mathcal{P}$ the following three statements are equivalent [13]:

(**c1**) For every two elements a and b from $\mathcal{P}$ there exist $c, d \in \mathcal{P}$ such that $d \leq a \leq c$, $d \leq b \leq c$ and $c \ominus a = b \ominus d$;

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(**c2**) For every two elements a and b from $\mathcal{P}$ there exists $c \in \mathcal{P}$ such that $a \le c$, $b \le c$ and $c \ominus a \le b$ (equivalently, $c \ominus b \le a$);
(**c3**) For every two elements a and b from $\mathcal{P}$ there exists $d \in \mathcal{P}$ such that $d \le a$, $d \le b$ and $b \ominus d \le 1 \ominus a$ (equivalently, $a \ominus d \le 1 \ominus b$).

D-lattices [6] and D-lattices of mutually compatible elements (so-called Boolean D-posets [5]) are noncommutative generalizations of Boolean algebras and in the realm of D-posets they play a particularly important role.

A D-lattice is a D-poset which is a lattice. D-lattices and D-lattices of mutually compatible elements can be characterized as a D-posets with a totally defined difference operation on it.

A D-poset $\mathcal{P}$ is a D-lattice if and only if the difference $b - a$ is defined for all a and b from $\mathcal{P}$ and the following properties (DL1)–(DL4) are fulfilled:

(DL1) $a - 0 = a$ for every $a \in \mathcal{P}$;
(DL2) $a, b \in \mathcal{P}$, $a \le b$ implies $c - b \le c - a$ for every $c \in \mathcal{P}$;
(DL3) $b - (b - a) = a - (a - b)$ for every $a, b \in \mathcal{P}$;
(DL4) $a \le b \le c$ implies $(c - a) - (c - b) = b - a$.

The lattice operations in $\mathcal{P}$ are defined by:

$$a \wedge b = b - (b - a) = a - (a - b), \tag{1}$$

$$\begin{aligned} a \vee b &= 1 - [(1 - a) - ((1 - a) - (1 - b))] \\ &= 1 - [(1 - b) - ((1 - b) - (1 - a))]. \end{aligned} \tag{2}$$

A D-lattice $\mathcal{P}$ is a Boolean D-poset if condition (DL4) is replaced by a stronger one:

(DL4*) $(c - b) - a = (c - a) - b$ for every $a, b, c \in \mathcal{P}$.

Boolean D-posets are in literature known as Φ-symmetric effect algebras introduced later in [1], and they are term equivalent with MV-algebras introduced by Chang ([3]).

In a Boolean D-poset, operations $\odot$ and $\oplus$ are defined by the following formulas:

$$a \odot b = a - (1 - b) = b - (1 - a), \tag{3}$$

$$a \oplus b = 1 - [(1 - a) - b] = 1 - [(1 - b) - a]. \tag{4}$$

Basic properties of Boolean D-posets needed in this article have been introduced in [5]. Next we shall prove some additional properties.

Proposition 1 *Let a, b, c be elements of a Boolean D-poset $\mathcal{P}$. Then*

(i) $a \vee b = 1 - ((1 - b) - (a - b)) = 1 - ((1 - a) - (b - a))$;
(ii) $a - b = (a \vee b) - b = a - (a \wedge b)$;
(iii) $c - (a \vee b) = (c - a) \wedge (c - b)$;
(iv) $(a \wedge b) - c = (a - c) \wedge (b - c)$.

Proof (i)

$$\begin{aligned} (a \vee b) &= 1 - ((1 - a) \wedge (1 - b)) \\ &= 1 - [(1 - b) - ((1 - b) - (1 - a))] \end{aligned}$$

$$= 1 - [(1-b) - ((1-(1-a)) - b)]$$
$$= 1 - [(1-b) - (a-b)].$$

(ii) Using $a - b \leq 1 - b$,

$$\begin{aligned}(a \vee b) - b &= [1 - ((1-b) - (a-b))] - b \\ &= (1-b) - ((1-b) - (a-b)) \\ &= a - b.\end{aligned}$$

(iii) Because a Boolean D-poset is a distributive lattice, we get

$$\begin{aligned}c - (a \vee b) &= c - ((a \vee b) \wedge c) \\ &= c - ((a \wedge c) \vee (b \wedge c)) \\ &= (c - (a \wedge c)) \wedge (c - (b \wedge c)) \\ &= (c - a) \wedge (c - b).\end{aligned}$$

(iv)

$$\begin{aligned}(a \wedge b) - c &= [(1-(1-a)) \wedge (1-(1-b))] - c \\ &= [1 - ((1-a) \vee (1-b))] - c \\ &= (1-c) - ((1-a) \vee (1-b)) \\ &= ((1-c) - (1-a)) \wedge ((1-c) - (1-b)) \\ &= ((1-(1-a)) - c) \wedge ((1-(1-b)) - c) \\ &= (a-c) \wedge (b-c).\end{aligned}$$

□

From the point of view of applications in physics, the compatibility relation in D-posets [13] is rather important.

For D-posets, the compatibility of two elements a and b means their mutual relationship characterized by any of the equivalent conditions (c1)–(c3). Precisely, the compatibility of two elements implies the existence of two elements c and d, fulfilling the condition (c1), such that d is a joint segment of a and b, and c contains a and b. The elements c and d are not, unlike the classical case, defined uniquely.

It is well known that each orthomodular lattice of pairwise compatible elements forms a Boolean algebra. A similar result was obtained for orthomodular posets, where instead of the pairwise compatibility a stronger relation of the so-called f-compatibility has to be used.

For D-lattices we have a similar situation. In this case, a Boolean algebra is replaced by a Boolean D-poset. So, in a Boolean D-poset, each two elements a, b are compatible and the elements c and d in the definition of compatibility fulfill the following inequalities:

$$a \odot b = a - (1-b) \leq d \leq a - (a-b) = a \wedge b,$$
$$a \vee b = 1 - ((1-a) - (b-a)) \leq c \leq 1 - ((1-a) - b) = a \oplus b.$$

We remark that a Boolean D-poset $\mathcal{P}$ is a Boolean algebra if and only if, for every $a \in \mathcal{P}$, $a \wedge (1-a) = 0$.

A product on an MV-algebra $\mathcal{M}$ [18] is a commutative and associative binary operation on $\mathcal{M}$ such that, for all $a, b, c \in \mathcal{M}$, the following conditions are satisfied:

(P1) $1 \cdot a = a$;
(P2) $c \cdot (b - a) = c \cdot b - c \cdot a$.

The product on an MV-algebra $\mathcal{M}$ has the following properties:

(pro1) $0 \cdot a = 0$ *for all* $a \in \mathcal{M}$;
(pro2) *If* $a \leq b$ *then* $a \cdot c \leq b \cdot c$ *for all* $c \in \mathcal{M}$;
(pro3) $a - (a \cdot b) \leq 1 - b$;
(pro4) $a \odot b \leq a \cdot b \leq a \wedge b$;
(pro5) *If* $a - (1 - b) = 0$ *then* $c \cdot (a \oplus b) = (c \cdot a) \oplus (c \cdot b)$.

For each two elements a, b in an MV-algebra $\mathcal{M}$, their product $a \cdot b$ yields an element d, $d \leq a, b$, such that $a - d \leq 1 - b$. So $a \cdot b$ is one of the elements d from the definition of the compatibility of a and b.

The MV-algebras with product were studied also by Riečan [17], Dvurečenskij and Riečan [8], Di Nola and Dvurečenskij [7] (here a representation of Boolean D-posets with unital ℓ-rings was established), and Mundici [16] (here a representation of a product σ-complete MV-algebras by a clan with a pointwise product was proved).

2 Quasi-Product on Boolean D-Posets

A comparison of the properties of product on an MV-algebra and the properties of the compatible elements a, b leads to a generalization of the product operation. We define a binary operation in such a way that the product is its special case.

Definition 2 Let $(\mathcal{P}, \leq, 0, 1, -)$ be a Boolean D-poset. Let $\overline{\wedge}$ be a commutative and associative binary operation on $\mathcal{P}$ such that, for all $a, b, c \in \mathcal{P}$, the following conditions are satisfied:

(qp1) $1 \overline{\wedge} a = a$;
(qp2) If $a \leq b$, then $a \overline{\wedge} c \leq b \overline{\wedge} c$;
(qp3) $a - (a \overline{\wedge} b) \leq 1 - b$.

Then $\overline{\wedge}$ is called a *quasi product* on $\mathcal{P}$.

Example 3 The product on an MV-algebra $\mathcal{M}$ is a quasi product on $\mathcal{M}$.

Example 4 Let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set $\mathcal{Z}$. Let χ_A, $(\chi_A \equiv A)$ be a characteristic function of a set $A \in \mathcal{Z}$. The binary operation $\overline{\wedge}$ on $\mathcal{S}$ defined by $A \overline{\wedge} B = \chi_A \cdot \chi_B$ is a quasi product on $\mathcal{S}$.

Example 5 Let $\mathcal{L}$ be a Boolean D-poset. Then the binary operations $\wedge$ and $\odot$ are quasi products on $\mathcal{L}$.

The notion of a quasi product has a strong connection to t-norms. We recall that a *t-norm* [12] is any function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that it is commutative, associative, nondecreasing in both arguments, and 1 is a neutral element for T, i.e., $T(1, x) = x$ for any $x \in [0, 1]$.

For example: $T_M(a,b) = \min\{a,b\}$, $T_W(a,b) = \min\{a,b\}$ if $\max\{a,b\} = 1$, otherwise $T_W(a,b) = 1$ (drastic t-norm), $T_L(a,b) = \max\{a+b-1, 0\}$ (Łukasiewicz t-norm), $T_P(a,b) = ab$ (product t-norm) are examples of t-norms.

For any t-norm T we have $T_W \le T \le T_M$, and

$$T_L \le T_P \le T_M.$$

To each t-norm, T, we can assign its dual *t-conorm*, S, by $S(a,b) := 1 - T(1-a, 1-b)$. Then S is commutative, associative, 0 is a neutral element and S is nondecreasing in both components. A special class of t-norms are Frank t-norms, i.e. those that satisfy the identity

$$T(a,b) + S(a,b) = a + b \tag{5}$$

which is interesting for our study, see Definition 13 and Remark 14. Due to [12], the solutions to this equation form the family of Frank t-norms, $\{T^F_\lambda : \lambda \in [0,\infty]\}$, where $T^F_0 = T_M$, $T^F_1 = T_P$, $T^F_\infty = T_L$, and $T^F_\lambda = \log_\lambda(1 + (\lambda^a - 1)(\lambda^b - 1)/(\lambda - 1))$ if $\lambda \neq 0, 1, \infty$. Moreover,

$$T^F_\lambda \ge T_L.$$

The following two examples are due to Proposition 10.

Example 6 If we take the standard Boolean D-poset $\mathcal{P} = [0,1]$ of real numbers, where $a \odot b = \max\{a+b-1, 0\} = T_L(a,b)$, we see that any t-norm T which satisfies $T_L(a,b) \le T(a,b) \le T_M(a,b)$ gives an example of a quasi product on the standard Boolean D-poset $[0,1]$. In particular, if $T = T^F_\lambda$ for $\lambda \in [0,\infty)$, we have uncountably many different quasi products on $[0,1]$.

Let T be a t-norm. A *T-clan* of fuzzy sets [2] is a system $\mathcal{S}$ of fuzzy sets on Ω such that (i) $1 \in \mathcal{S}$, (ii) if $f \in \mathcal{S}$ then $1 - f \in \mathcal{S}$, and (iii) if $f, g \in \mathcal{S}$ then $T(f,g) \in \mathcal{S}$, where $T(f,g)$ is the fuzzy intersection obtained by the pointwise application of T, i.e., $T(f,g)(\omega) = T(f(\omega), g(\omega))$ for all $\omega \in \Omega$. In the particular case of $\mathcal{S} = [0,1]^\Omega$, we call $\mathcal{S}$ a *full tribe* [17].

Example 7 Let T be a t-norm from Example 6 and T_L be the Łukasiewicz t-norm. If $\mathcal{P} = [0,1]^\Omega$, or if $\mathcal{P} \subseteq [0,1]^\Omega$ is a T_L-clan which is also a T-clan, then $f \mathbin{\overline{\wedge}} g := T(f,g)$ is a quasi product on $\mathcal{P}$.

Example 8 Let $\mathcal{C}$ be the Chang MV-algebra, i.e., [9, Theorem 5.3.19], the elements of $\mathcal{C}$ are $0 < 1 < 2 < \cdots < 2^* < 1^* < 0^*$. For $m, n \in \{0, 1, 2, \ldots\}$, we define $n \mathbin{\overline{\wedge}} m = 0$, $n \mathbin{\overline{\wedge}} m^* = m^* \mathbin{\overline{\wedge}} n = \max\{n - m, 0\}$, and $n^* \mathbin{\overline{\wedge}} m^* = (n+m)^*$. Then we have a Boolean D-poset which is not isomorphic to any clan of fuzzy sets, with a nontrivial quasi product $\overline{\wedge}$ different from $\odot$.

In what follows, we present some basic properties of the quasi product on Boolean D-posets.

Proposition 9 *Let $\mathcal{P}$ be a Boolean D-poset with a quasi product $\overline{\wedge}$ and let $a, b \in \mathcal{P}$. Then*

(i) $a \mathbin{\overline{\wedge}} b \le a$, $a \mathbin{\overline{\wedge}} b \le b$;
(ii) $a \mathbin{\overline{\wedge}} 0 = 0$.

Proof A straightforward corollary of the properties (qp1) and (qp2) in Definition 2. □

Proposition 10 *Let $\mathcal{P}$ be a Boolean D-poset with a quasi product $\overline{\wedge}$. Then*

$$a \odot b \leq a \,\overline{\wedge}\, b \leq a \wedge b \tag{6}$$

for every $a, b \in \mathcal{P}$.

Proof We recall that in this case the difference operation is total and $b - a = b - (a \wedge b)$. From the property (qp3) in Definition 2 and from the properties of a difference operation we have

$$a - (1 - b) \leq a - (a - (a \,\overline{\wedge}\, b)) = a \,\overline{\wedge}\, b. \tag{7}$$

From the property (i) in Proposition 9 we get $a \,\overline{\wedge}\, b \leq a \wedge b$. □

Theorem 11 *Let $\mathcal{L}$ be a Boolean D-poset and let $a, b \in \mathcal{L}$. The quasi products $\odot$ and $\wedge$ on $\mathcal{L}$ satisfy the following conditions*:

(i) $(b - a) \odot c \leq (b \odot c) - (a \odot c)$;
(ii) $(b - a) \wedge c \geq (b \wedge c) - (a \wedge c)$.

Proof Let $a, b, c \in \mathcal{L},\ a \leq b$.

(i) Then $b - a \leq b - (a - (1 - c))$ and by (3) we have

$$\begin{aligned}(b - a) \odot c &= (b - a) - (1 - c)\\ &= [b - (1 - c)] - a\\ &\leq (b - (1 - c)) - (a - (1 - c))\\ &= (b \odot c) - (a \odot c).\end{aligned}$$

(ii) From $b \geq b \wedge c$ we get

$$\begin{aligned}b - a &\geq (b \wedge c) - a\\ &= (b \wedge c) - (b \wedge c \wedge a)\\ &= (b \wedge c) - (c \wedge a).\end{aligned}$$

Because $(b \wedge c) - (c \wedge a) \leq c$, the inequality

$$\begin{aligned}(b - a) \wedge c &\geq ((b \wedge c) - (c \wedge a)) \wedge c\\ &= (b \wedge c) - (c \wedge a)\end{aligned}$$

holds. □

Now we introduce a new operation, $\underline{\vee}$, which is in some sense dual to $\overline{\wedge}$, and it will resemble (5) for Frank t-norms.

Theorem 12 *Let $\mathcal{P}$ be a Boolean D-poset with a quasi product $\bar{\wedge}$. Define a binary operation $\veebar$ on $\mathcal{P}$ by*

$$a \veebar b = 1 - ((1-b) - (a - (a \bar{\wedge} b))), \tag{8}$$

$a, b \in \mathcal{P}$. Then

(i) $a \veebar b = 1 - ((1-a) - (b - (a \bar{\wedge} b)))$;
(ii) $a \veebar b \geq a,\ a \veebar b \geq b$;
(iii) $(a \veebar b) - a = b - (a \bar{\wedge} b)$, $(a \veebar b) - b = a - (a \bar{\wedge} b)$,

and $\veebar$ is commutative.

Proof To prove (i), we start from the expression $(1-b) - (a - (a \bar{\wedge} b))$. Then

$$\begin{aligned}(1-b) - (a - (a \bar{\wedge} b)) &= (1-b) - ((1 - (a \bar{\wedge} b)) - (1-a))\\ &= (1-b) - [((1 - (a \bar{\wedge} b)) - (b - (a \bar{\wedge} b)))\\ &\quad -((1-a) - (b - (a \bar{\wedge} b)))]\\ &= (1-b) - [(1-b) - ((1-a) - (b - (a \bar{\wedge} b)))]\\ &= (1-a) - (b - (a \bar{\wedge} b)).\end{aligned}$$

Hence

$$1 - ((1-b) - (a - (a \bar{\wedge} b))) = 1 - ((1-a) - (b - (a \bar{\wedge} b))).$$

(ii) From $(1-a) - (b - (a \bar{\wedge} b)) \leq 1 - a$ we get

$$a = 1 - (1-a) \leq 1 - ((1-a) - (b - (a \bar{\wedge} b))) = (a \veebar b).$$

The proof of $a \veebar b \geq b$ is similar and it is omitted.

(iii)

$$\begin{aligned}(a \veebar b) - a &= [1 - ((1-a) - (b - (a \bar{\wedge} b)))] - a\\ &= (1-a) - ((1-a) - (b - (a \bar{\wedge} b)))\\ &= b - (a \bar{\wedge} b).\end{aligned}$$

The proof of $(a \veebar b) - b = a - (a \bar{\wedge} b)$ is similar and it is omitted. The commutativity follows from (i). □

Definition 13 Let $\mathcal{P}$ be a Boolean D-poset with a quasi product $\bar{\wedge}$. Then $\veebar$ is called a *quasi-sum* on $\mathcal{P}$.

It is worthy recalling that in view of Mundici's representation of MV-algebras by intervals in unital Abelian ℓ-groups, [15], the operation "$-$" taken in the Boolean D-poset (= MV-algebra) coincides with "$-$" taken in groups. Therefore, by (iii) of Theorem 12, we obtain an analogue for (5) with group addition $+$.

Remark 14 If $\bar{\wedge}$ is the t-norm from Example 6, then the corresponding quasi sum is $a \veebar b = a + b - a \bar{\wedge} b$. This is *not* the t-conorm dual to $\bar{\wedge}$, it even need not be monotonic. If $\bar{\wedge}$ is a Frank t-norm, [12], then $\veebar$ is the t-conorm dual to $\bar{\wedge}$, and hence is monotonic.

Proposition 15 *Let $\mathcal{P}$ be a Boolean D-poset. Let u be an arbitrary element of $\mathcal{P}$. Define a binary operation $\overline{\wedge}_u$ on $\mathcal{P}$ by*

$$a \,\overline{\wedge}_u\, b = a - ((a \vee u) - b), \tag{9}$$

$a, b \in \mathcal{P}$. Then

(i) $a \,\overline{\wedge}_u\, b = b \,\overline{\wedge}_u\, a$;
(ii) $a \,\overline{\wedge}_u\, b = a \wedge (b - (u - a)) = b \wedge (a - (u - b))$;
(iii) $(a \,\overline{\wedge}_u\, b) \,\overline{\wedge}_u\, c = a \,\overline{\wedge}_u\, (b \,\overline{\wedge}_u\, c)$;
(iv) $1 \,\overline{\wedge}_u\, a = a$;
(v) *If* $a \leq b$ *then* $a \,\overline{\wedge}_u\, c \leq b \,\overline{\wedge}_u\, c$;
(vi) $a - (a \,\overline{\wedge}_u\, b) \leq 1 - b$.

Proof In the proof we use the properties of Boolean D-posets introduced in [5] and in Proposition 1.

(i)

$$\begin{aligned} a \,\overline{\wedge}_u\, b &= a - ((a \vee u) - b) \\ &= a - ((a \vee u \vee b) - b) \\ &= [(a \vee u \vee b) - ((a \vee u \vee b) - a)] - ((a \vee u \vee b) - b) \\ &= [(a \vee u \vee b) - ((a \vee u \vee b) - b)] - ((a \vee u \vee b) - a) \\ &= b - ((u \vee b) - a) \\ &= b \,\overline{\wedge}_u\, a. \end{aligned}$$

(ii)

$$\begin{aligned} a \,\overline{\wedge}_u\, b &= a - ((a \vee u) - b) \\ &= a - ((a - b) \vee (u - b)) \\ &= (a - (a - b)) \wedge (a - (u - b)) \\ &= (a \wedge b) \wedge (a - (u - b)) \\ &= b \wedge (a - (u - b)). \end{aligned}$$

(iii) First we prove the associativity. Using (i) and (ii) we get

$$\begin{aligned} (a \,\overline{\wedge}_u\, b) \,\overline{\wedge}_u\, c &= [a \wedge (b - (u - a))] \,\overline{\wedge}_u\, c \\ &= c \wedge ([a \wedge (b - (u - a))] - (u - c)) \\ &= c \wedge [(a - (u - c)) \wedge [(b - (u - a)) - (u - c)]] \\ &= a \wedge (c - (u - a)) \wedge [(b - (u - c)) - (u - a)]. \end{aligned}$$

$$\begin{aligned} a \,\overline{\wedge}_u\, (b \,\overline{\wedge}_u\, c) &= a \,\overline{\wedge}_u\, [c \wedge (b - (u - c))] \\ &= a \wedge ([c \wedge (b - (u - c)] - (u - a)) \\ &= a \wedge (c - (u - a)) \wedge [(b - (u - c)) - (u - a)]. \end{aligned}$$

(iv) $1 \,\bar{\wedge}_u\, a = 1 - ((1 \vee u) - a) = 1 - (1 - a) = a$.
(v) Let $a \leq b$. Then $(c \vee u) - b \leq (c \vee u) - a$ and so $c - ((c \vee u) - a) \leq c - ((c \vee u) - b)$.
(vi)

$$\begin{aligned} a - (a \,\bar{\wedge}_u\, b) &= a - [a - ((a \vee u) - b)] \\ &= ((a \vee u) - b) - [((a \vee u) - b) - a] \\ &\leq (a \vee u) - b \\ &\leq 1 - b. \end{aligned}$$

□

The next theorem is an immediate corollary of the previous Proposition 15.

Theorem 16 *Let $\mathcal{P}$ be a Boolean D-poset. Then the binary operation defined by* (9) *is a quasi product on $\mathcal{P}$.*

Example 17 If we apply Proposition 15 to Example 6, we obtain a t-norm $\bar{\wedge}_u$ with one ordinal summand which is the Łukasiewicz t-norm on the interval $[0, u]$, i.e., $a \,\bar{\wedge}_u\, b = \max\{a + b - u, 0\}$ if $a, b \in [0, u]$, $a \,\bar{\wedge}_u\, b = \min\{a, b\}$ otherwise. As in Example 7, this example can be easily generalized to T_L-clans ($\odot$-clans) which are also T-clans ($\bar{\wedge}_u$-clans).

Example 18 Let us apply Proposition 15 to Example 8.

One class of examples is obtained for $u \in \{1, 2, 3, \ldots\}$. Then $a \,\bar{\wedge}_u\, b = \max\{a + b - u, 0\}$ if $a, b \leq u$, $a \,\bar{\wedge}_u\, b = \min\{a, b\}$ otherwise.

Another class of examples is obtained for $u = k^*$, $k \in \{1, 2, 3, \ldots\}$. For elements $a \in \{m, m^*\}$, $b \in \{n, n^*\}$, where $m, n \in \{0, 1, 2, \ldots\}$ the quasi product $\bar{\wedge}_u$ is defined by the following rules: If $a \geq u$ or $b \geq u$, then $a \,\bar{\wedge}_u\, b = \min\{a, b\}$. In the remaining cases, $n \,\bar{\wedge}\, m = 0$, $n \,\bar{\wedge}\, m^* = m^* \,\bar{\wedge}\, n = \max\{n - m + k, 0\}$, and $n^* \,\bar{\wedge}\, m^* = (n + m - k)^*$.

These examples are Boolean D-posets which are not isomorphic to any clan of fuzzy sets and admit a nontrivial quasi product $\bar{\wedge}_u$ different from $\odot$.

3 Conclusion

We believe that the quasi product on Boolean D-posets will be a suitable tool for further study of mutual relationships between the elements of a Boolean D-poset.

From the algebraic point of view it would be interesting to investigate further properties of Boolean D-posets with a quasi product, e.g., in connection with Theorem 16.

Boolean D-posets or MV-algebras, respectively, are axiomatic models of non-Kolmogorovian (non–commutative) probability theory. Elements of a Boolean D-poset represent events of some probability space for which we do not have enough information to obtain a Boolean algebra. The elements $a \,\underline{\vee}\, b$ and $a \,\bar{\wedge}\, b$ represent the union and the intersection corresponding to events in this probability space.

In the probability theory on Boolean D-posets, states and observables are basic notions. States on MV-algebras have been studied for example in [4, 11]. We recall that a state on a Boolean D-poset $\mathcal{P}$ is any mapping $m : \mathcal{P} \to [0, 1]$ such that $m(1) = 1$ and $m(b - a) = m(b) - m(a)$ whenever $a \leq b$. From the point of view of states, property (iii) in Theorem 12 is relevant. Indeed, if m is a state on a Boolean D-poset $\mathcal{P}$ with a quasi product $\bar{\wedge}$, then $m(a \,\underline{\vee}\, b) + m(a \,\bar{\wedge}\, b) = m(a) + m(b)$.

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